MATH 121A Prep: Proofs

Facts to Know:

Induction: Proof technique for showing a statement is true for all for all positive integers

- Base Case: Prove true for $n=1$ (smallest value you care about)
- Inductive Step: If true for n, its also true for $n+1$.

Quantifiers:

- There exists: $\exists$ true for something in the set
- For all: $\forall$ true for every single element in the set

Negation of quantifiers:

- Negation of there exists: negate $\exists$ "statement" $\to$ $\forall$ "negation statement"
- Negation of for all: negate $\forall$ statement $\to$ $\exists$ negation statement

Examples:

1. Let $A = \begin{bmatrix} a & b-a \\ 0 & b \end{bmatrix}$. Prove the formula $A^n = \begin{bmatrix} a^n & b^n - a^n \\ 0 & b^n \end{bmatrix}$ show this equality holds for every $n \geq 1$.

Base Case: $n=1$

$$A^1 = A = \begin{bmatrix} a & b-a \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^1 & b^1-a^1 \\ 0 & b^1 \end{bmatrix},$$

Induction Step: Assume $A^n = \begin{bmatrix} a^n & b^n - a^n \\ 0 & b^n \end{bmatrix}$.

$$\overset{?}{=} b^{n+1} - a^{n+1}$$

$$A^{n+1} = AA^n = \begin{bmatrix} a & b-a \\ 0 & b \end{bmatrix} \begin{bmatrix} a^n & b^n-a^n \\ 0 & b^n \end{bmatrix} = \begin{bmatrix} a^{n+1} & a(b^n-a^n) + (b-a)b^n \\ 0 & b^{n+1} \end{bmatrix}$$

$$a(b^n-a^n)+(b-a)b^n = \cancel{ab^n} - a^{n+1} + b^{n+1} - \cancel{ab^n} = b^{n+1} - a^{n+1}!$$

Therefore this is true for all $n \geq 1$.

2. Convert the following statements between words and mathematical notation:

   (a) For all objects y in a set Y there exists an x in X such that $f(x)$ equals y.

   $\forall y \in Y \ \exists x \in X$ such that (s.t.) $f(x) = y$.

   (b) $\exists \vec{v_1}, \vec{v_2} \in \mathbb{R}^2$ such that $\forall \vec{w} \in \mathbb{R}^2$ $\exists!\, c_1, c_2 \in \mathbb{R}$ where $\vec{w} = c_1\vec{v_1} + c_2\vec{v_2}$

   there exists vectors $\vec{v}_1, \vec{v}_2$ in $\mathbb{R}^2$ such that for all $\vec{w}$ in $\mathbb{R}^2$ there exists <u>unique</u> c_1, c_2 in $\mathbb{R}$ where $\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2$.

3. Write the negation of the above statements:

   (a) There exists y in Y such that for all $x \in X$ we have $f(x) \neq y$.

   (b) $\forall\ \vec{v}_1, \vec{v}_2 \in \mathbb{R}$, $\exists \vec{w} \in \mathbb{R}^2$, $\forall c_1, c_2 \in \mathbb{R}$ then $\vec{w} \neq c_1\vec{v}_1 + c_2\vec{v}_2$.